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Dimension-2 condensates, ζ-regularization and large-N_c Regge models

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Abstract. Dimension-2 and -4 gluon condensates are re-analyzed in large-N_c Regge models with the ζ-function regularization which preserves the spectrum in any $\bar{q}q$ channel separately. We demonstrate that the signs and magnitudes of both condensates can be properly described within the framework.

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The dimension-2 gluon condensate, corresponding to the vacuum expectation value of a gauge-invariant non-perturbative and non-local operator and generating the lowest $1/Q^2$ power corrections, was proposed long ago [1] and has been determined in instanton model studies [2], phenomenological QCD sum rule re-analyses [3,4], theoretical considerations [5–9], non-local quark models [10], and lattice simulations at zero [11,12] and finite [13] temperatures. In the present paper we confirm our recent findings [14], namely, that within the large-N_c expansion these $1/Q^2$ corrections appear naturally within the Regge framework, in the light of ζ-regularization. We show that the signs and magnitudes of *both* the dimension-2 and -4 condensates can be accommodated comfortably with reasonable values of the parameters of the hadronic spectra. Many works compare Regge models to the Operator Product Expansion (OPE) [15–19] but besides [19] the dimension-2 condensate has been ignored.

We begin with a simple quantum-mechanical derivation of the (radial) Regge spectrum. For two relativistic scalar quarks of mass m interacting via a linear confining potential the mass operator in the CM frame is given by

$$M = 2\sqrt{\boldsymbol{p}^2 + m^2} + \sigma_S r, \qquad (1)$$

where σ_S is the (scalar) string tension and $\boldsymbol{p}$ and r are the relative momentum and distance, respectively. Squaring $\sqrt{\boldsymbol{p}^2 + m^2}$ yields an equivalent Schrödinger operator, such that $\boldsymbol{p}^2 = p_r^2 + L^2/r^2$. For excited radial states the Bohr-Sommerfeld semiclassical quantization condition holds,

$$2\pi(n + \alpha) = 2\int_0^a p_r(r)\mathrm{d}r, \qquad (2)$$

where $p_r(r)$ is the local classical radial momentum defined by $p_r(r)^2 + L^2/r^2 + m^2 = (M - \sigma_S r)^2/4$, a is the turning point and α is of order of unity. Taking $L = 0$ and $m = 0$ for simplicity, one has $p_r(r) = |M - \sigma_S r|/2$ and $a = M/\sigma_S$. The integral is trivial leading to

$$M_n^2 = 4\pi\sigma_S(n + \alpha) = 2\pi\sigma(n + \alpha), \qquad (3)$$

a (radial) Regge mass spectrum which in terms of the spinor string tension $\sigma = 2\sigma_S$ (the factor depends on the type of interaction [20]) seems fulfilled experimentally for mesons [21] and signals confinement for the quarks. For large meson masses, the level density becomes

$$\rho(M^2) = \sum_n \delta(M^2 - M_n^2) \to \int \mathrm{d}n \delta(M^2 - M_n^2)$$
$$= \frac{\mathrm{d}n}{\mathrm{d}M^2} = \frac{1}{\pi} \int_0^a \frac{\mathrm{d}r}{p_r(r)} = \frac{1}{2\pi\sigma} \qquad (4)$$

which is a constant. Inclusion of finite quark mass corrections is straightforward, yielding

$$\frac{\mathrm{d}n}{\mathrm{d}M^2} = \frac{1}{2\pi\sigma}\sqrt{1 - \frac{4m^2}{M^2}} \quad \text{for} \quad M^2 \geq 4m^2. \qquad (5)$$

Note that this corresponds to the two-body phase space factor appearing in the absorptive part of two-point correlators. Thus, at large energies the WKB approximation holds and $\rho(M^2)$ looks like the phase space of two free particles, featuring the *quark-hadron duality*.

The best way to look at the $q\bar{q}$ level density is to consider two-point correlation functions in different channels,

$$i\Pi^{\mu a, \nu b}(q) = \int \mathrm{d}^4 x e^{-iq \cdot x} \langle 0|T\left\{J^{\mu a}(x)J^{\nu b}(0)\right\}|0\rangle$$
$$= \Pi(q^2)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\delta^{ab}, \qquad (6)$$

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where current conservation for both the vector and axial currents has explicitly been used. At high Euclidean momentum OPE can be performed. Equivalently, in the Euclidean coordinate space one has

$$\langle J_\mu(x) J_\nu(0) \rangle = \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \right) \Pi(x). \tag{7}$$

The function $\Pi(x)$ has dimension x^{-4}. Thus, at short distances one expects to have (up to possible logarithms)

$$\Pi(x) = \frac{O_0}{x^4} + \frac{O_2}{x^2} + \frac{O_4}{x^0} + O_6 x^2 + \ldots. \tag{8}$$

We digress that the term containing O_2, an operator of dimension-2, is very special, since it yields a contribution of the form $(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)x^{-2} = (g^{\mu\nu}x^2 - 4x^\mu x^\nu)x^{-4}$ which, in addition to being conserved, is also traceless. Thus, contracting and taking the derivative ∂^α do no commute. There is no traceless and transverse term in momentum space, since conservation implies the form $A(q^\mu q^\nu - q^2 g^{\mu\nu})$ but tracelessness requires $A = 0$. This problem appears in chiral quark models; the dimension-2 object is the constituent-quark mass squared, $O_2 \sim \langle \sigma^2 + \pi^2 \rangle \sim M^2$.

Recent discussions incorporate O_2 in OPE. From [9] we get up to dimension 6 in the chiral limit,

$$\Pi_{V+A}(Q^2) = \frac{1}{4\pi^2}\left\{ -\left(1 + \frac{\alpha_S}{\pi}\right)\log\frac{Q^2}{\mu^2} \right.$$
$$\left. -\frac{\alpha_S}{\pi}\frac{\lambda^2}{Q^2} + \frac{\pi}{3}\frac{\langle\alpha_S G^2\rangle}{Q^4} + \frac{256\pi^3}{81}\frac{\alpha_S\langle\bar{q}q\rangle^2}{Q^6} \right\},$$
$$\Pi_{V-A}(Q^2) = -\frac{32\pi}{9}\frac{\alpha_S\langle\bar{q}q\rangle^2}{Q^6}, \tag{9}$$

where λ^2 corresponds to the dimension-2 condensate.

In the large-N_c limit one has (up to subtractions) [22]

$$\Pi_V(Q^2) = \sum_V \frac{F_V^2}{M_V^2 + Q^2} + \text{c.t.}, \tag{10}$$

where the sum involves infinitely many resonances. This function satisfies a dispersion relation of the form

$$\Pi_V(Q^2) = \int_0^\infty \mathrm{d}s \frac{Q^2}{s}\frac{\rho_V(s)}{s + Q^2}, \tag{11}$$

with one subtraction, $\Pi_V(0) = 0$, due to Coulomb's law and the spectral function in the vector channel given by

$$\rho_V(s) = \frac{1}{\pi}\operatorname{Im}\Pi_V(s) = \sum_V F_V^2 \delta(s - M_V^2). \tag{12}$$

At large values of the squared CM energy s, it becomes

$$\rho_V(s) \to \int_0^\infty F_V^2 \delta(s - M_V^2)\mathrm{d}n = F_V(n)^2 \frac{\mathrm{d}n}{\mathrm{d}M_V^2}\Big|_{M_V^2 = s}. \tag{13}$$

Matching to the free massless quark result $\rho_V(s) = N_c/(12\pi^2)\theta(s)$ gives at large values of n

$$F_V(n)^2 \frac{\mathrm{d}n}{\mathrm{d}M_V^2} \to \frac{N_c}{3}\frac{1}{4\pi^2}. \tag{14}$$

For constant F_V this implies the asymptotic spectrum $M_V^2 \to 2\pi\sigma n$ with the string tension given by $2\pi\sigma = 24\pi^2 F_V^2/N_c$. If we identify $F_V = 154\,\text{MeV}$ from the $\rho \to 2\pi$ decay [23] which corresponds to only one resonance, we get $\sqrt{\sigma} = 546\,\text{MeV}$. Lattice calculations [24] provide $\sqrt{\sigma} = 420\,\text{MeV}$.

In dimensional regularization the coupling of the resonance to the current acquires an additional dimension $F_V \to F_V\mu^\epsilon$ with $\epsilon = d - 4$. By choosing $\mu = M_V$ one gets $F_V^2 \to F_V^2 M_V^{2\epsilon}$. The regularized correlator is an analytic function for $Q^2 < m_\rho^2$ (the lowest mass), so we can Taylor-expand at small Q^2. We can then regularize the finite coefficients of the expansion and proceed by analytic continuation both in Q^2 and ϵ. The regularization only acts for truly *infinitely* many resonances. At large Euclidean momenta one gets

$$\Pi_V(Q^2) = \sum_V F_V^2 M_V^{2\epsilon} - \sum_V F_V^2 \frac{M_V^{2+2\epsilon}}{Q^2} + \ldots. \tag{15}$$

The coefficients of powers in $1/Q^2$ of the expansion are convergent provided one computes the sum first and then takes the limit $\epsilon \to 0$ corresponding to the use of the ζ-function regularization (see, *e.g.*, [25]),

$$\sum_V F_V^2 M_V^{2n} \equiv \lim_{s\to n}\sum_V F_V^2 M_V^{2s}. \tag{16}$$

In other words, one may expand formally at large Q^2 and re-interpret the result by means of the ζ-function regularization. Using the axial-axial correlator at large N_c,

$$\Pi_A(Q^2) = \frac{f^2}{Q^2} + \sum_A \frac{F_A^2}{M_A^2 + Q^2} + \text{c.t.}, \tag{17}$$

and matching to (9) yields the two Weinberg sum rules:

$$f^2 = \sum_A F_A^2 - \sum_V F_V^2 \qquad\qquad \text{(WSR I)},$$

$$0 = \sum_A F_A^2 M_A^2 - \sum_V F_V^2 M_V^2 \qquad \text{(WSR II)}.$$

These sums are assumed to be ζ-regularized, see eq. (16).

The simplest Regge model is given by

$$M_{V,n}^2 = M_V^2 + 2\pi\sigma n, \qquad M_{A,n}^2 = M_A^2 + 2\pi\sigma n, \tag{18}$$

$n = 0, 1, 2\ldots$, which seems fulfilled [21] experimentally. The corresponding couplings are constant, $F_V = F_A = F$ and the ζ-function regularized sums follow from

$$\sum_{n=0}^\infty (na + M^2)^s = a^s \zeta\left(-s, \frac{M^2}{a}\right). \tag{19}$$

This function is analytic in the complex plane $\operatorname{Re}(s) \leq -1$ with the exception of $s = -1$, admitting analytic continuation to any s. Actually, for positive powers one gets the Bernoulli polynomials $\zeta(-k, z) = -B_{k+1}(z)/(k+1)$,
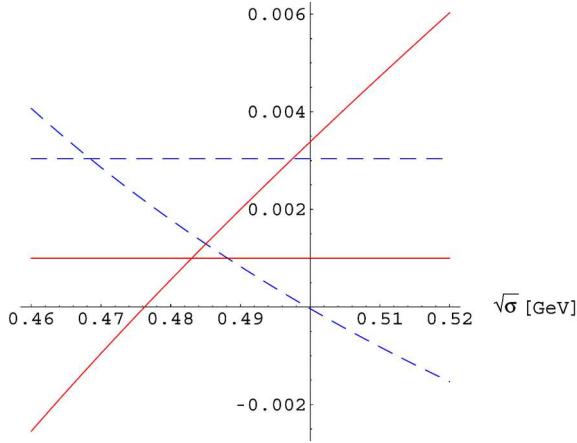
Fig. 1. Dimension-2 (solid line, in GeV^2) and -4 (dashed line, in GeV^4) condensates plotted as functions of $\sqrt{\sigma}$. The horizontal lines indicate estimates from the literature.

where $B_0 = 1$, $B_1 = x - 1/2$, $B_2 = x^2 - x + 1/6$, etc. An important feature of the ζ-function is that it regulates each spectrum separately, *i.e.* under regularization one cannot apply the distributive property. For instance,

$$\sum_{n=0}^{\infty} (an + M_V^2)^0 - \sum_{n=0}^{\infty} (an + M_A^2)^0 \neq 0. \qquad (20)$$

In other words, the difference of the regularized sums does not coincide with the regularized difference. The finite terms in the difference have to do with preserving the spectra in the vector and axial channels separately, and hence a chiral asymmetry is generated. All these ζ-function results reproduce the direct asymptotic expansion of exact sums in terms of the digamma functions, but allow to discuss cases where the sums cannot be carried out *before* expanding in large Q^2.

The strict linear Regge model does not generate condensates with the proper signs. In [14] (see also [26]) we consider the following simple modification:

$$M_{V,0} = m_\rho, \ M_{V,n}^2 = M_V^2 + 2\pi\sigma n, \quad n \geq 1,$$
$$M_{A,n}^2 = M_A^2 + 2\pi\sigma n, \quad n \geq 0. \qquad (21)$$

Here the lowest ρ mass is shifted, otherwise all is kept "universal", including constant residues for all states. With (21) the Weinberg sum rules are (we set $N_c = 3$)

$$M_A^2 = M_V^2 + 8\pi^2 f^2,$$
$$2\pi\sigma = 8\pi^2 F^2 = \frac{8\pi^2 f^2 \left(4\pi^2 f^2 + M_V^2\right)}{4\pi^2 f^2 - m_\rho^2 + M_V^2}. \qquad (22)$$

When $m_\rho = 0.77$ GeV is fixed, the model has only one free parameter left. We may take it to be M_V; however, it is more convenient to express it through the string tension σ, which is then treated as a free parameter. Thus

$$M_V^2 = \frac{-16\pi^3 f^4 + 4\pi^2 \sigma f^2 - m_\rho^2 \sigma}{4 f^2 \pi - \sigma}, \qquad (23)$$

and the condensates, obtained by matching to (9), are

$$-\frac{\alpha_S \lambda^2}{4\pi^3} = \frac{16\pi^3 f^4 - \pi\sigma^2 + m_\rho^2 \sigma}{16 f^2 \pi^3 - 4\pi^2 \sigma}$$

$$\frac{\alpha_S \langle G^2 \rangle}{12\pi} = 2\pi^2 f^4 - \pi\sigma f^2$$
$$+ \frac{3\sigma m_\rho^2}{8\pi^2} \left(\frac{m_\rho^2 \sigma}{(\sigma - 4 f^2 \pi)^2} - 2\pi \right) + \frac{\sigma^2}{12}. \qquad (24)$$

Despite the strong dependence on $\sqrt{\sigma}$ (see fig. 1) the correct signs, $-\lambda^2 > 0$, $\langle G^2 \rangle > 0$, and reasonable numbers are reproduced in the range $0.48 \leq \sqrt{\sigma} \leq 0.5$ GeV [14].

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